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Linear systems with randomly interrupted Gaussian white noise

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Abstract. A linear process driven by additive Gaussian white noise, which is randomly interrupted by an exponentially correlated two-state $\{0, 1\}$ Markovian stochastic process, is considered. A characteristic function of the process is obtained using an approach based on conditional functionals for Markov processes. A single-event time dependent probability distribution is presented. Steady states are analysed in terms of stationary distributions and moments of the process. The deviation from Gaussianity (kurtosis) is investigated.

1. Introduction

In a narrow sense, a diffusion process is determined by a second-order partial differential equation of parabolic type. In a wide sense, any evolution influenced by stochastic fluctuations, random perturbations or noises is named a diffusion process. A mathematical study of random evolution of the system begins with the modelling of stochastic perturbations. The simplest model is Gaussian white noise [1] with zero correlation time. It describes parabolic diffusion (because it is determined by differential equations of parabolic type). Non-Gaussian white noises, in particular Poisson white noise [2, 3] and correlated Gaussian noises [4, 5] have been receiving considerations in the literature. Non-Gaussian correlated processes have been applied as well [6]. In particular, an exponentially correlated two-state Markov process describes the so-called hyperbolic diffusion [7] determined by a second-order partial differential equation of hyperbolic type.

In this paper we shall consider processes driven by randomly interrupted Gaussian white noise [8, 9]. The random interruption is realized by an exponentially correlated two-state Markov process which can take two possible values $\{0, 1\}$. The value 0 means that the white noise is switched off and the value 1 corresponds to switching on the white noise. This kind of process can be a caricature of random media [8] or it can represent a controlled noise generated in the laboratory by specific devices and introduced in the system of interest in order to investigate its influences and reaction of the system.

In section 2 we present general considerations on a system with interrupted noise. An evolution equation of the integro-differential type for a one-dimensional probability distribution function is presented in this section, too. In section 3 we consider a particular case of the model, namely, a linear system with additive noise. This model is exactly soluble. To obtain a probability distribution of the process, a conditional functional of the two-state Markov process is introduced in section 4. This functional satisfies a set of two ordinary differential equations which are solved in section 4. In section 5, a time-evolution of the probability distribution is considered and some special cases (in dependence of values of the correlation time of the two-state Markov process) are presented. In section 6 we study stationary distributions. Section 7 is devoted to moments of the process of interest.

2. General considerations

In a previous paper [9], we considered processes driven by randomly interrupted Gaussian noises and described by a stochastic equation (in the Stratonovich sense [1]):

$$\dot{x}_t = f(x_t) + g(x_t)\gamma(t) \qquad x \in (x_1, x_2) \tag{2.1}$$

for a given relevant variable x_t. The functions $f(x)$ and $g(x)$ are deterministic, $\gamma(t)$ is a stochastic process defined as [8, 9]

$$\gamma(t) = (1/2)[1 + \xi(t)]\eta(t) \tag{2.2}$$

where $\eta(t)$ is Gaussian white noise [1]

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(s) \rangle = 2D\delta(t-s) \qquad D > 0 \tag{2.3}$$

and $\xi(t) = \{-1, 1\}$ is dichotomic noise (a symmetric two-state Markov process) [10]

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(s) \rangle = \exp(-2\nu|t-s|). \tag{2.4}$$

Transition probabilities $+1 \to -1$ and $-1 \to +1$ in small time dt equal νdt. Initial probabilities for $\xi(t)$ are chosen as

$$p_1(0) \equiv \Pr\{\xi(0) = 1\} = \tfrac{1}{2} \qquad p_{-1}(0) \equiv \Pr\{\xi(0) = -1\} = \tfrac{1}{2}. \tag{2.5}$$

It is assumed that an initial value x_0 of the process (2.1) is statistically independent of $\gamma(t)$. It is also assumed that $\eta(t)$ and $\xi(t)$ are statistically independent of each other. Hence

$$\langle \gamma(t) \rangle = 0 \qquad \langle \gamma(t)\gamma(s) \rangle = D\delta(t-s). \tag{2.6}$$

The *non-Gaussian* noise $\gamma(t)$ is independent of the system, its intensity is D and it has zero correlation time (white noise!). The higher-order correlation functions of $\gamma(t)$ can be obtained by making use of the Gaussian character of $\eta(t)$ and properties of $\xi(t)$ [2].

If the variable t is time then $\gamma(t)$ can be interpreted as a Langevin force switched on and off at random instants. If t is a spatial variable then $\gamma(t)$ can model stochastic two-layer medium: one layer is a medium with a diffusion coefficient $D_1 = D$ [8, 11] and the other is a vacuum (surrounding characterized by a diffusion coefficient $D_2 = 0$). It is the simplest model of randomly stratified media and can be starting point of generalizations for N layers with different diffusion coefficients D_i ($i = 1, 2, \ldots, N$) for each layer [11].

As possible applications of equations like (2.1) one can mention the problem of multiple scattering of particles through plates of matter separated by vacuum gaps [8]; transport phenomena in sponge-type structures with empty places (vacua) and matter randomly distributed in space; wave propagation in randomly stratified media [12], and so on. Equation (2.1) is a particular case of two-state models in which transitions from one state (determinstic: $\dot{x}_t = f(x_t)$) to the other state (diffusional: $\dot{x}_t = f(x_t) + g(x_t)\eta(t)$) and vice versa occur at random time. Such models are mentioned by van Kampen [13]. Balakrishnan *et al* [14] considered first-passage time problems for processes like (2.1). Indeed, their equation (3.1) in [14] reduces to our equation (2.1) when $f_+(x) = f_-(x) = f(x)$, $g_-(x) = 0$ and $g_+(x) = g(x)$.

In [9] it was shown that the time-evolution of a single-event probability distribution $P(x, t)$ of the process (2.1) satisfies the following integro-differential equation

$$\begin{aligned}\frac{\partial}{\partial t} P(x,t) = &-\frac{\partial}{\partial x} f(x) P(x,t) + \frac{D}{2}\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) P(x,t) \\ &+ \frac{D^2}{4}\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \int_0^t \mathrm{d}s\, \mathrm{e}^{-2\nu(t-s)} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathcal{G}(x,t|y,s) \\ &\times \frac{\partial}{\partial y} g(y) \frac{\partial}{\partial y} g(y) P(y,s)\end{aligned} \tag{2.7}$$

where $\mathcal{G}(x, t|y, s)$, $t \geqslant s$, is a transition probability distribution of a diffusion process determined by the infinitesimal generator $\hat{B}(x)$:

$$\hat{B}(x) = -\frac{\partial}{\partial x} f(x) + \frac{D}{2}\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x). \tag{2.8}$$

Notice that the first two terms in the right-hand side of equation (2.7) represent the standard parabolic diffusion whereas the remainder describes effects connected with the correlation time $\tau_c \equiv 1/2\nu$ of the process $\xi(t)$. Indeed, if the correlation time τ_c tends to zero ($\nu \to \infty$) then the last term in the right-hand side of equation (2.7) tends to zero and (2.7) reduces to a Fokker–Planck equation.

3. Linear processes with additive noise

It is obvious that equation (2.7) cannot be solved for the general case. Therefore it is worth investigating simple cases for which $P(x, t)$ can exactly be calculated. The first case (pure diffusive), when $f(x) = 0$ and $g(x) = 1$, was studied by Laskin [8]. Here, we wish to consider a second fundamental model, a linear system with additive noise $\gamma(t)$:

$$\dot{x}(t) = -ax(t) + \gamma(t) \qquad x \in (-\infty, \infty) \qquad a > 0. \tag{3.1}$$

Linear equations like (3.1) can describe relaxation processes, the decay of unstable states, a position of an overdamped particle in a harmonic potential (when inertial effects can be neglected), a particle's velocity in the absence of an external potential and so on.

Our aim is to determine the probability distribution $P(x, t)$ of the process (3.1) with the initial condition

$$P(x, 0) = \rho(x) \tag{3.2}$$

where $\rho(x)$ is given. We could start with equation (2.7). The transition probability $\mathcal{G}(x, t|y, s)$ is known in this case because then (2.8)) is a generator of the

Ornstein–Uhlenbeck process [1]. Unfortunately, even in this simple case it is rather difficult to solve equation (2.7). Therefore we will calculate $P(x, t)$ in another way. Notice that

$$P(x, t) = \int_{-\infty}^{\infty} p(x, t|x_0, 0)\rho(x_0)\, dx_0 \tag{3.3}$$

where $p(x, t|x_0, 0)$ is an initial transition probability distribution of the process (3.1) which can be expressed as

$$p(x, t|x_0, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega x}\, C(\omega, t) \tag{3.4}$$

where $C(\omega, t)$ is a characteristic function of the process (3.1)

$$C(\omega, t) = \langle e^{i\omega x(t)} \rangle^{\xi, \eta} \tag{3.5}$$

with $x(t)$ being a solution of (3.1) for a given realization of the process $y(t)$. The superscripts ξ and η indicate average over all realizations of the processes $\xi(t)$ and $\eta(t)$, respectively (for notational convenience we have dropped explicit dependence of $C(\omega, t)$ on the initial value x_0 of $x(t)$). The procedure of averaging over all realizations of the Gaussian process $\eta(t)$ is well known [1] and gives

$$C(\omega, t) = \exp\left[i\omega x_0 e^{-at} - \frac{\omega^2 D}{4a}(1 - e^{-2at}) \right] \tilde{C}(\omega, t) \tag{3.6}$$

where

$$\tilde{C}(\omega, t) = \left\langle \exp\left[-\frac{1}{2}\omega^2 D\, e^{-2at} \int_0^t ds\, e^{2as}\xi(s) \right] \right\rangle^{\xi}. \tag{3.7}$$

To find $\tilde{C}(\omega, t)$ we will follow a procedure similar to that used in [15].

4. Conditional functional

Consider the functional $\mathbb{F}[\xi|\Omega, T]$ of the stochastic process $\xi(t)$ defined as follows

$$\mathbb{F}[\xi|\Omega, T] = \left\langle \exp\left[-\Omega \int_0^T ds\, e^{2as}\xi(s) \right] \right\rangle^{\xi}. \tag{4.1}$$

For fixed time $t = T$ and for

$$\Omega = \tfrac{1}{2}\omega^2 D\, e^{-2aT} \tag{4.2}$$

one has [15]

$$\tilde{C}(\omega, t) = \mathbb{F}[\xi|\Omega = \tfrac{1}{2}\omega^2 D\, e^{-2at}, t]. \tag{4.3}$$

Let us introduce the conditional functional $W(t, b)$ defined by the relation [16]

$$W(t, b) = \left\langle \exp\left[-\Omega \int_0^t ds\, e^{2as}\xi(s)\right] | \xi(0) = b \right\rangle^{\xi} \tag{4.4}$$

where $\langle A|B\rangle^{\xi}$ denotes a mean value over ξ of the quantity A under the condition B and $b=\{-1, 1\}$ (for notational convenience we drop the explicit functional dependence of $W(t, b)$ upon the process $\xi(t)$).

The functional $\mathbb{F}[\xi|\Omega, T]$ can be obtained from (4.4) by averaging $W(t, b)$ over all possible initial values of $\xi(t)$, namely

$$\mathbb{F}[\xi|\Omega, T]=\langle W(T, b)\rangle^{\xi(0)}=\sum_{b=-1,1} W(T, b)p_b(0). \tag{4.5}$$

The initial probabilities $p_b(0)$ are given in (2.5) and therefore (4.5) takes the form

$$\mathbb{F}[\xi|\Omega, T]=\tfrac{1}{2}[W(T, +1)+W(T, -1)]. \tag{4.6}$$

It can be shown [16] that the conditional functional (4.4) obeys the following set of equations

$$\frac{\partial}{\partial t} W(t, b)=LW(t, b)-b\Omega\, e^{2\alpha t}W(t, b) \tag{4.7}$$

where $W(t, b)$ is a column vector built from $W(t, 1)$ and $W(t, -1)$. The operator L is a Kolmogorov backward operator for the process $\xi(t)$ and has the form[1] [16]

$$L\doteq\begin{pmatrix} -\nu & \nu \\ \nu & -\nu \end{pmatrix}. \tag{4.8}$$

The initial conditions for equations (4.7) follow from (4.4) and read

$$W(0, 1)=1 \qquad W(0, -1)=1. \tag{4.9}$$

Equations (4.7) form a system of two linear non-autonomous ordinary differential equations. To solve them, let us define a new variable τ as

$$\tau=\tau(t)=\Omega\, e^{2\alpha t} \tag{4.10}$$

and a new function $V(\tau, b)$ via the relation

$$W(t, b)=V(\tau(t), b). \tag{4.11}$$

Then the set (4.7) can be converted into equations

$$2\alpha\tau\frac{\partial}{\partial\tau} V(\tau, 1)=-\nu V(\tau, 1)+\nu V(\tau, -1)-\tau V(\tau, 1) \tag{4.12a}$$

$$2\alpha\tau\frac{\partial}{\partial\tau} V(\tau, -1)=\nu V(\tau, 1)-\nu V(\tau, -1)+\tau V(\tau, -1) \tag{4.12b}$$

with the initial conditions

$$V(\Omega, 1)=1 \qquad V(\Omega, -1)=1. \tag{4.13}$$

Defining the functions $G(\tau)$ and $F(\tau)$ as

$$G(\tau)=\tfrac{1}{2}[V(\tau, 1)-V(\tau, -1)] \tag{4.14a}$$

[1] In the general case, Kolmogorov backward and forward operators are different. In the case considered here, they are equal because the transition probabilities per unit time $1\to -1$ and $-1\to 1$ are equal.

$$F(\tau)=\tfrac{1}{2}[V(\tau,1)+V(\tau,-1)] \tag{4.14b}$$

and using (4.12) yield

$$\begin{aligned} &2aF'(\tau)=-G(\tau)\\ &2a\tau G'(\tau)=-2\omega G(\tau)-\tau F(\tau)\\ &F(\Omega)=1 \qquad G(\Omega)=0 \end{aligned} \tag{4.15}$$

(the prime denotes a derivative with respect to τ) from which we get the equation

$$a\tau F''(\tau)+\omega F'(\tau)-(\tau/4a)F(\tau)=0 \tag{4.16}$$

with the conditions

$$F(\Omega)=1 \qquad F'(\Omega)=0. \tag{4.17}$$

The solution of equation (4.16) is known [17] and reads

$$F(\tau)=\tau^{\mu}\left[A_1 I_{-\mu}\left(\frac{\tau}{2a}\right)+A_2 K_{-\mu}\left(\frac{\tau}{2a}\right)\right] \tag{4.18}$$

where

$$\mu=\frac{1}{2}\left(1-\frac{\nu}{a}\right). \tag{4.19}$$

The functions $I_\mu(z)$ and $K_\mu(z)$ are modified Bessel functions. Constants A_1 and A_2 can be obtained from the conditions (4.17)

$$A_1=\frac{K_{-\mu+1}(\Omega/2a)}{2a\Omega^{\mu-1}} \qquad A_2=\frac{I_{-\mu+1}(\Omega/2a)}{2a\Omega^{\mu-1}}. \tag{4.20}$$

One can take an arbitrary set of linearly independent solutions of the modified Bessel differential equation but (4.18) is useful for analysis of the long-time limit and stationary probabilities.

5. Time-dependent distribution

From equations (3.7), (4.3), (4.6), (4.11) and (4.14b) we obtain the relation

$$\tilde{C}(\omega,t)=F(\tau) \tag{5.1}$$

for

$$\tau=\tau(t)=\Omega e^{2at} \qquad \Omega=\tfrac{1}{2}\omega^2 D\,e^{-2at}. \tag{5.2}$$

Next, from (3.4), (3.6), (5.1), (5.2) and (4.18) we obtain the initial transition probability distribution $p(x,t|y,0)$ of the process (3.1) in the following form

$$p(x,t|y,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega x}\, e^{i\omega y e^{-at}} N(\omega,t) \tag{5.3}$$

where

$$\begin{aligned} N(\omega,t)=&\,\alpha\omega^2 e^{-(a+\nu)t}\exp[-\alpha\omega^2(1-e^{-2at})]\\ &\times[K_{-\mu+1}(\alpha\omega^2 e^{-2at})I_{-\mu}(\alpha\omega^2)+I_{-\mu+1}(\alpha\omega^2 e^{-2at})K_{-\mu}(\alpha\omega^2)] \end{aligned} \tag{5.4}$$

and

$$\alpha=D/4a. \tag{5.5}$$

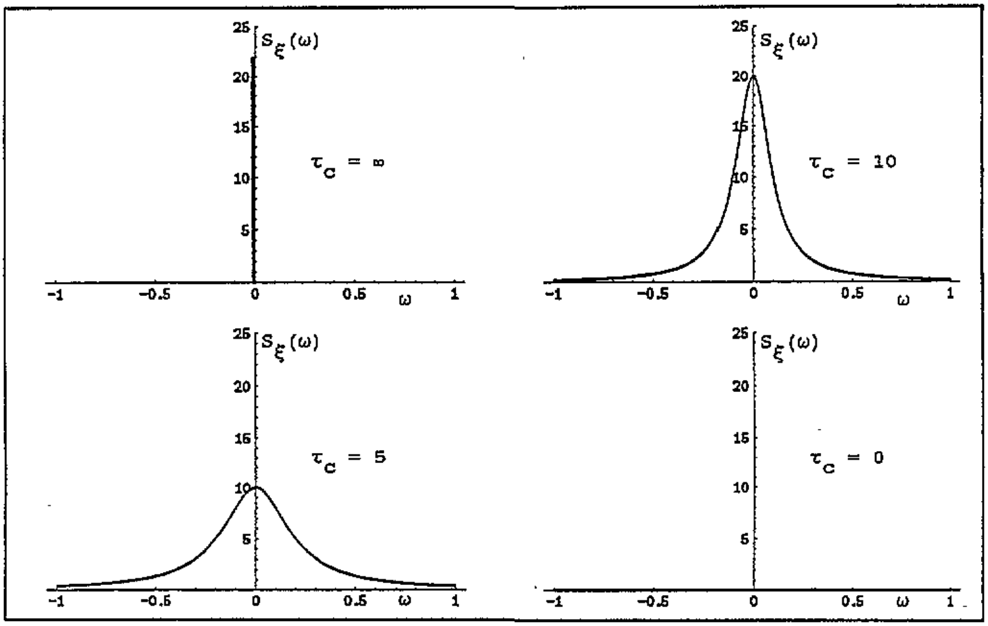

Figure 1. Some selected examples of the spectral density $S_\xi(\omega)$ of the process $\xi(t)$ for various values of the correlation time τ_c of $\xi(t)$. For $\tau_c = \infty$, it is a Dirac δ-function represented by a vertical line. For $\tau_c = 0$, $S_\xi(\omega) = 0$.

As $\rho(x)$ in (3.3) one can choose an arbitrary initial distribution for $x(t)$. If we assume the deterministic initial state, $\rho(x) = \delta(x - x_0)$, then $P(x, t)$ equals in fact $p(x, t|x_0, 0)$. Random initial states lead to no new essential properties of $P(x, t)$. In particular, a stationary distribution does not depend on the initial state. Therefore here we consider a case of the deterministic initial condition. Our model contains three parameters:

D; the intensity of the noise $\eta(t)$,
$\tau_c = 1/2\nu$; the correlation time of $\xi(t)$,
$\tau_d = 1/a$; the deterministic relaxation time of $x(t)$.

The crucial role plays the relation between τ_c and τ_d.

5.1. *The case of infinite* τ_c

Physically, this case corresponds to situations for which $\tau_c \gg \tau_d$. If τ_c is infinite then the process $\xi(t)$ is strongly correlated

$$\langle \xi(t)\xi(s)\rangle = 1 \tag{5.6}$$

for any t and s.

The spectral density $S_\xi(\omega)$ of the process $\xi(t)$ defined as

$$S_\xi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \langle \xi(t)\xi(t+\tau)\rangle \, d\tau \tag{5.7}$$

has for this case the form (figure 1)

$$S_\xi(\omega) = 2\pi\delta(\omega). \tag{5.8}$$

The infinite τ_c corresponds to $\mu=\frac{1}{2}$ in (4.19) and then the integrand in (5.3) can be expressed by elementary functions [17]. The integral in (5.3) can be calculated with the result

$$p(x,t|x_0,0)=\tfrac{1}{2}\delta(x-x_0\,\mathrm{e}^{-at}) + \frac{1}{2}\left[\frac{a}{2\pi D(1-\mathrm{e}^{-2at})}\right]^{1/2}\exp\left[-\frac{a(x-x_0\,\mathrm{e}^{-at})^2}{2D(1-\mathrm{e}^{-2at})}\right]. \tag{5.9}$$

This equation has an evident interpretation. If τ_c tends to infinity then the transition probabilities $1\to-1$ and $-1\to1$ of the process $\xi(t)$ tend to zero. With probability $\frac{1}{2}$ noise $\gamma(t)$ in (3.1) is switched off and then $x(t)$ is a deterministic process. It corresponds to the Dirac delta distribution in (5.9). With probability $\frac{1}{2}$ the noise is switched on and then $x(t)$ is a Gaussian process. It corresponds to the second term in the right-hand side of equation (5.9).

5.2. *The case* $0<\tau_c<\infty$

Physically, this case should be associated with situations when τ_c is of order τ_d. Now, $\xi(t)$ is exponentially correlated (2.4) and the spectral density has the Lorentzian form (figure 1)

$$S_\xi(\omega)=\frac{2\tau_c}{1+(\tau_c\omega)^2}. \tag{5.10}$$

In a general case, the integral in (5.3) is not calculable. Here, we present an example when $\tau_c=(1/4)\tau_d$ or $\nu=2a$ [18]:

$$p(x,t|x_0,0)=\tfrac{1}{2}\,\mathrm{e}^{-at}\delta(x-x_0\,\mathrm{e}^{-at}) + [2-\tfrac{3}{2}\,\mathrm{e}^{-2at}]\left[\frac{a}{2\pi D(1-\mathrm{e}^{-2at})}\right]^{1/2}\exp\left[-\frac{a(x-x_0\,\mathrm{e}^{-at})^2}{2D(1-\mathrm{e}^{-2at})}\right] - \frac{a}{D}\,|x-x_0\,\mathrm{e}^{-at}|\,\mathrm{erfc}\left[\left[\frac{a}{2D(1-\mathrm{e}^{-2at})}\right]^{1/2}|x-x_0\,\mathrm{e}^{-at}|\right]. \tag{5.11}$$

The time-evolution of (5.11) is visualized in figure 2.

5.3. *The case* $\tau_c=0$

Physically, this case corresponds to situations when $\tau_c\ll\tau_d$. For this case, $\xi(t)$ is uncorrelated

$$\langle\xi(t)\xi(s)\rangle=0 \qquad \text{for } t\neq s \tag{5.12}$$

(but $\langle\xi^2(t)\rangle=1$ for any time t) and its spectral density is zero (figure 1)

$$S_\xi(\omega)=0 \tag{5.13}$$

The process $x(t)$ reduces to an Ornstein–Uhlenbeck process [1] with

$$p(x,t|x_0,0)=\left[\frac{a}{\pi D(1-\mathrm{e}^{-2at})}\right]^{1/2}\exp\left[-\frac{a(x-x_0\,\mathrm{e}^{-at})^2}{D(1-\mathrm{e}^{-2at})}\right]. \tag{5.14}$$

Let us notice that although Gaussian functions in (5.14) and (5.9) are different, fluctuations determined by them have the same strength.

6. Stationary distribution

The stationary distribution $P(x)$ can be found by calculating the limit $t \to \infty$ in (5.3). To obtain $P(x)$, let us express modified Bessel functions in (5.4) as [17]

$$I_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu+1)} e^{-z} {}_1F_1(\mu + \tfrac{1}{2}, 2\mu+1, 2z)$$
$$K_\mu(z) = \pi^{1/2}(2z)^\mu e^{-z} U(\mu + \tfrac{1}{2}, 2\mu+1, 2z) \tag{6.1}$$

where ${}_1F_1(\beta, \gamma, z)$ is a Kummer (confluent hypergeometric) function and $U(\beta, \gamma, z)$ is a Tricomi function [17].

From (6.1) it follows that

$$\lim_{t\to\infty} I_{-\mu+1}(\alpha\omega^2 e^{-2at}) = 0 \tag{6.2}$$

and

$$\lim_{t\to\infty} \alpha\omega^2 e^{-(a+\nu)t} K_{-\mu+1}(\alpha\omega^2 e^{-2at}) = \left(\frac{\alpha\omega^2}{2}\right)^\mu \Gamma(1-\mu) \tag{6.3}$$

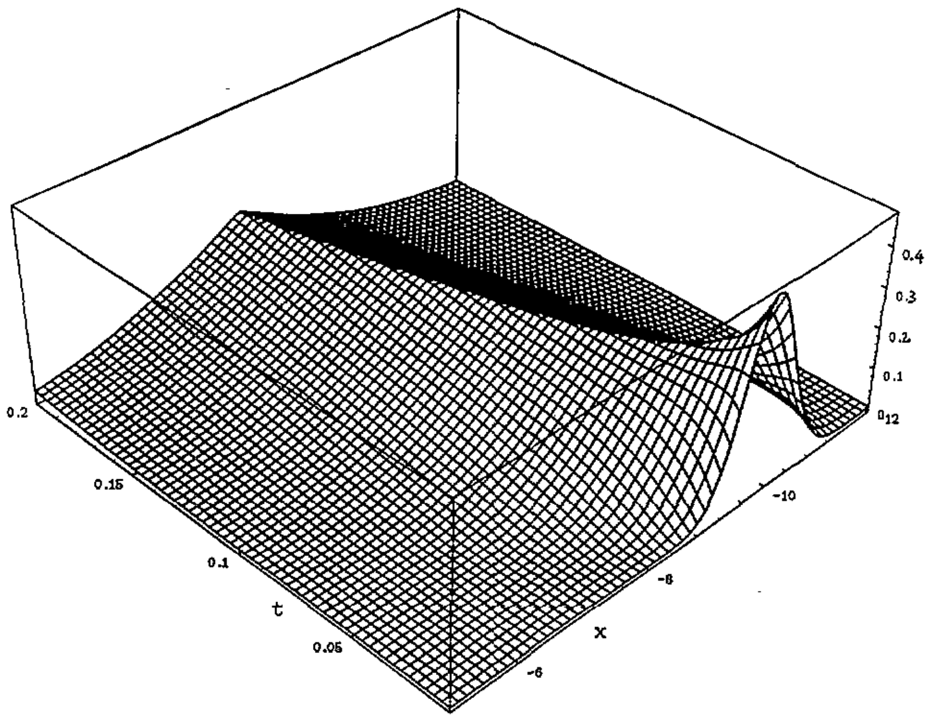

Figure 2. Graph showing a part of the distribution $p(x, t|x_0, 0)$ (3.8) for $D/a = 10$, $\tau_c = (1/4)\tau_d$ and $x_0 = -10$. For a better visualization, the Dirac δ-part of $p(x, t|x_0, 0)$ is not presented.

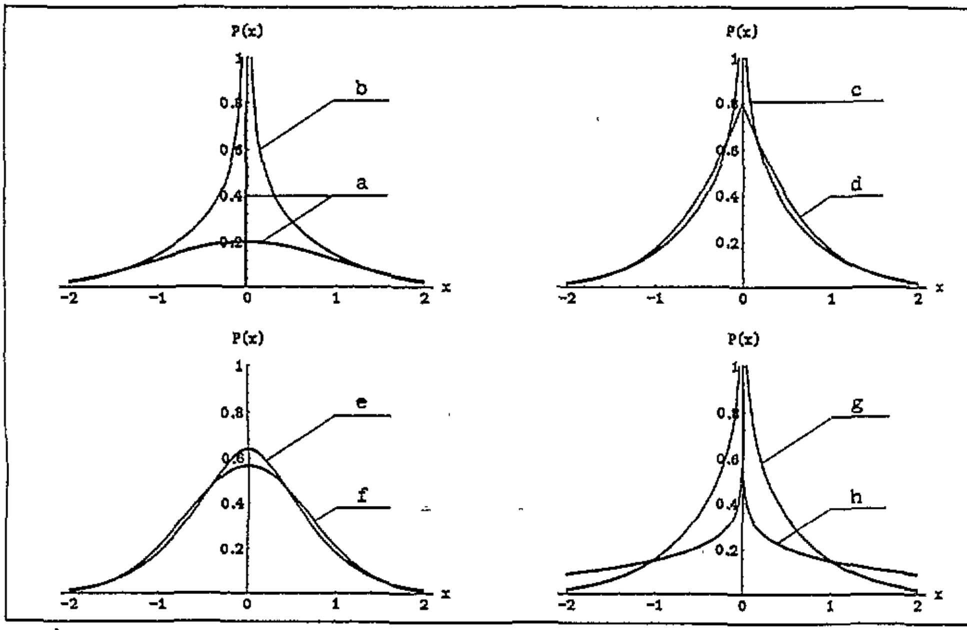

Figure 3. Plots of the stationary probability distribution $P(x)$ versus x for (a) $\tau_c = \infty$, (b) $\tau_c = \tau_d \equiv 1/a$, (c) $\tau_c = (1/2)\tau_d$, (d) $\tau_c = (1/4)\tau_d$, (e) $\tau_c = (1/8)\tau_d$, (f) $\tau_c = 0$ (Gaussian) and $D/a = 1$. The curves: (g) $D/a = 1$, (h) $D/a = 10$ and for $\tau_c = (1/2)\tau_d$.

where $\Gamma(z)$ stands for an Euler gamma function.

Taking into account (6.1)–(6.3) in (5.3) and (5.4) allows to present the stationary probability distribution in the form

$$P(x) = (2^{-\mu}/\pi)\Gamma(1-\mu)\int_0^\infty d\omega\,\cos(\omega x)\,(a\omega^2)^\mu\,e^{-a\omega^2}I_{-\mu}(a\omega^2). \tag{6.4}$$

The expression (6.4) represents in general three classes of distributions:

(i) singular distributions;
(ii) functions which diverge to infinity when the argument tends to zero;
(iii) functions which are finite at zero.

6.1. *The case of infinite τ_c*

The case of a singular distribution is realized when the correlation time τ_c of the process $\xi(t)$ is infinite. Then the parameter $\mu = \frac{1}{2}$ and from (6.4) we get (figure 3)

$$P(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\sqrt{\frac{a}{2\pi D}}\exp\left(-\frac{ax^2}{2D}\right). \tag{6.5}$$

If follows also from equation (5.9) in the limit $t \to \infty$.

6.2. *The case of finite* τ_c

For $\tau_c < \infty$ ($\mu < \frac{1}{2}$) the integration in (6.4) can be carried out [19]. As a result we obtain

$$P(x) = \frac{\Gamma(1-\mu)}{\pi}\left(\frac{\alpha}{2}\right)^{\mu/2}(x^2)^{-(\mu+1)/2}\exp\left(-\frac{x^2}{16\alpha}\right)W_{3\mu/2,\,-\mu/2}\left(\frac{x^2}{8\alpha}\right)$$

where $W_{\kappa,\nu}(z)$ stands for a Whittaker function [17].

6.2.1. $\tau_d \leqslant 2\tau_c < \infty$. In this case, $P(x)$ diverges to infinity when x tends to zero: e.g. if $\tau_c = \tau_d$ then $P(x)$ can be expressed by elementary functions (figure 3)

$$P(x) = (2\pi)^{-1}\Gamma(3/4)\sqrt{a/D}\left(\frac{ax^2}{2D}\right)^{-1/4}\exp\left(-\frac{ax^2}{2D}\right). \tag{6.7}$$

For $x \to 0$ it diverges as $|x|^{-1/2}$.

If $\tau_c = (1/2)\tau_d$ then (figure 3)

$$P(x) = \frac{1}{\pi}\sqrt{\frac{a}{2\pi D}}\exp\left(-\frac{ax^2}{4D}\right)K_0\left(\frac{ax^2}{4D}\right). \tag{6.8}$$

It has the logarithmic divergence [17], $K_0(x) \cong -\gamma - \ln(x/2)$ for $0 < x \ll 1$.

6.2.2. $0 \leqslant 2\tau_c < \tau_d$. In this case, $P(x)$ has no singularities at $x = 0$: e.g., if $\tau_c = (1/4)\tau_d$ then (figure 3)

$$P(x) = \sqrt{\frac{2a}{\pi D}}\exp\left(-\frac{ax^2}{2D}\right) - \frac{a}{D}|x|\operatorname{erfc}\left(\sqrt{\frac{ax^2}{2D}}\right) \tag{6.9}$$

For $\tau_c = 0$ the process $x(t)$ becomes Gaussian and its stationary distribution can be inferred form (5.14).

7. Stationary moments

All stationary moments $\langle x^k \rangle$, $k = 1, 2, 3, \ldots$, are equal to zero for odd k. For even k they can be calculated using (6.6) and then [20]

$$\langle x^{2n} \rangle = \frac{\Gamma(1-\mu)}{\pi}2^{n-2\mu}\left(\frac{D}{a}\right)^n \frac{\Gamma(\frac{1}{2}+n-\mu)\Gamma(\frac{1}{2}+n)}{\Gamma(1+n-2\mu)} \tag{7.1}$$

for $n = 1, 2, 3, \ldots$.

The second moment $\langle x^2 \rangle$ does not depend on the correlation time τ_c of the process $\xi(t)$ and reads

$$\langle x^2 \rangle = D/2a. \tag{7.2}$$

The moments of higher order depend on τ_c. In particular (cf figure 4)

$$\langle x^4 \rangle = 3\frac{4a\tau_c + 1}{2a\tau_c + 1}\langle x^2 \rangle^2 \tag{7.3}$$

$$\langle x^6 \rangle = 15\frac{8a\tau_c + 1}{2a\tau_c + 1}\langle x^2 \rangle^3 \tag{7.4}$$

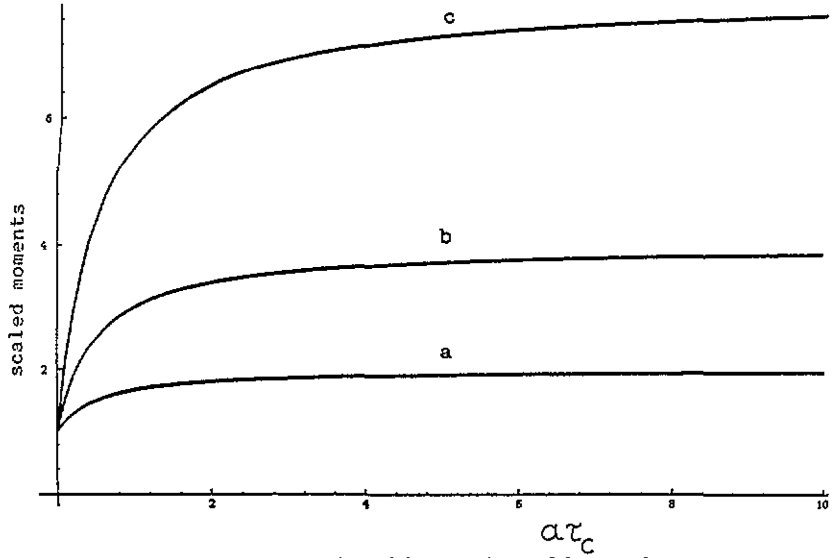

Figure 4. Scaled moments: (a) $\langle x^4\rangle/3\langle x^2\rangle^2$; (b) $\langle x^6\rangle/15\langle x^2\rangle^3$; (c) $\langle x^8\rangle/105\langle x^2\rangle^4$. The case $a\tau_c = 0$ corresponds to Gaussian statistics.

$$\langle x^8\rangle = 105\frac{(12a\tau_c+1)(8a\tau_c+1)}{(6a\tau_c+1)(2a\tau_c+1)}\langle x^2\rangle^4. \tag{7.5}$$

Having moments, one can construct some characteristics of distribution $P(x)$. The skewness [21] characterizes the degree of symmetry of $P(x)$ and in our case it is equal to zero (it means that $P(x)$ is symmetrical with respect to $x \to -x$). The kurtosis [21] measures peakedness or flatness of $P(x)$ in relation to the Gaussian distribution. In the case considered, the kurtosis is given by

$$\mathrm{Kurt}(x) = \frac{\langle x^4\rangle}{\langle x^2\rangle^2} - 3 = \frac{6a\tau_c}{2a\tau_c+1} \tag{7.6}$$

and is always positive, $0 \leqslant \mathrm{Kurt}(x) \leqslant 3$. It means that the stationary distribution $P(x)$ is *leptokurtic* [21].

In the limit $\tau_c \to \infty$, the moments $\langle x^{2n}\rangle = 2^{n-1}\langle x^{2n}\rangle_G$ (the subscript G denotes Gaussian statistics) and they are greater than Gaussian moments. In the limit $\tau_c \to 0$, all moments (7.1) reduce to moments for a corresponding Gaussian process and kurtosis is zero.

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